Anomalous diffusion of particles driven by correlated noise

S. I. Denisov*
Department of Mechanics and Mathematics, Sumy State University, 2, Rimskiy-Korsakov Street, 40007 Sumy, Ukraine

Werner Horsthemke†
Department of Chemistry, Southern Methodist University, Dallas, Texas 75275-0314
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We study the effect of an arbitrary stationary random force on the motion of damped particles. Using a Langevin description, we derive exact expressions for the dispersion of the particle position, of the particle velocity, and their cross dispersion. The particles can exhibit anomalous diffusion, and the connection between this behavior and the functional form of the noise correlations is investigated in detail. We also study anomalous diffusion for the special cases of overdamped and undamped particles.

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I. INTRODUCTION

Random processes play an important role in many fields of natural science [1]. A wide class of such processes gives rise to normal diffusion, where the mean-square displacement $\sigma_x^2(t)$ increases linearly with t for long times. Many processes, however, are characterized by anomalous diffusion where $\sigma_x^2(t) \sim t^\nu$ with $\nu \neq 1$ [2–4]. Following the conventional terminology, we will call the case $\nu > 1$ superdiffusion (diffusion faster than the normal) and the case $\nu < 1$ subdiffusion (diffusion slower than the normal). Superdiffusion is encountered, for example, in turbulent fluids [5], chaotic systems [6], layered velocity fields [7], or rotating flows [8], and subdiffusion in disordered ionic chains [9], porous systems [10], amorphous semiconductors [11,12], or disordered materials [13]. Anomalous diffusion is often caused by memory effects and Lévy-type statistics [2,3]. Specifically, superdiffusion is observed for random walks with long-tail jump-length distributions, and subdiffusion for long-tail waiting-time distributions. The latter type of distributions can be caused by ''traps'' that have an infinite mean waiting time [2].

Anomalous diffusion has been described by fractional diffusion equations [14–17], nonlinear Fokker-Planck equations [18–21], fractional Fokker-Planck equations [22–25], and different types of Langevin equations [26–34]. The Langevin equation is an attractive starting point for the treatment of diffusive behavior. It provides an equation of motion for the particles and accounts for the dynamical origins of diffusive motion. The classical Langevin equation describes the regular diffusion of Brownian particles [35]. The Langevin method is especially informative when those equations can be solved exactly. One can then represent the main characteristics of the solutions by quadratures, find their long-time asymptotics, study their dependence on the statistical characteristics of the driving noise, etc. The availability of such analytical expressions is advantageous in that it provides exact relationships between the driving forces and the diffusive behavior. The Langevin method has mostly been used to study anomalous diffusion of free particles without dissipation [27,30,31], with time-dependent friction [34], and with nonlocal dissipation, described by a generalized Langevin equation with a friction memory kernel [26,28,32,33].

Our aim is to show that ordinary Langevin dynamics can account for anomalous diffusion in nonequilibrium systems. We study the statistical properties of damped particles governed by the equation of motion

$$m\ddot{x}(t) + \lambda\dot{x}(t) = f(t), \tag{1.1}$$

where $x(t)$ is the particle position, m is the mass of the particle, and λ is a damping coefficient. The random force $f(t)$ is stationary (in the wide sense), has zero mean and the arbitrary correlation function

$$\langle f(t)f(t')\rangle = r(|t-t'|) = r(u). \tag{1.2}$$

Since we focus on nonequilibrium systems where the driving force $f(t)$ represents external noise, the damping coefficient λ and the correlations of the random force are not related to each other by a fluctuation-dissipation theorem. Our exact results can easily be specialized to the case of internal equilibrium fluctuations, and we comment on this fact in the appropriate places. If $f(t)$ is Gaussian white noise, i.e., $r(u) \sim \delta(u)$, where $\delta(u)$ is the Dirac δ function, then Eq. (1.1) describes the motion of so-called Rayleigh particles [1], which display normal diffusive behavior. Our main goal is to elucidate how correlations of the noise affect the statistical characteristics of $x(t)$ and, specifically, what properties of $r(u)$ are responsible for superdiffusive behavior of $x(t)$, and which ones for subdiffusive behavior.

The paper is organized as follows. In Sec. II, we derive exact expressions for the dispersion of the particle position, of the particle velocity, and their cross dispersion. Their long-time asymptotics are obtained in Sec. III, where we also discuss the conditions for superdiffusion and subdiffusion and consider the limits of undamped and overdamped particles. Two examples of specific correlation functions are studied in Sec. IV. Concluding remarks are contained in Sec. V.

*Electronic address: denisov@ssu.sumy.ua
†Electronic address: whorsthe@mail.smu.edu

II. GENERAL ANALYSIS

The Langevin equation (1.1) with the initial conditions

$$x(0)=0, \quad \dot{x}(0)=0, \tag{2.1}$$

has the solution

$$x(t)=\frac{e^{-\gamma t}}{m}\int_0^t dt'' e^{\gamma t''}\int_0^{t''} dt' f(t'), \tag{2.2}$$

where $\gamma=\lambda/m$. According to Eqs. (2.2) and (1.2), the dispersion of the particle position, $\sigma_x^2(t)=\langle x^2(t)\rangle$, is given by

$$\sigma_x^2(t)=\frac{e^{-2\gamma t}}{m^2}\int_0^t dt'' e^{\gamma t''}\int_0^t dt_1'' S(t'',t_1'') e^{\gamma t_1''}, \tag{2.3}$$

where

$$S(t'',t_1'')=\int_0^{t''} dt' \int_0^{t_1''} dt_1' r(|t'-t_1'|). \tag{2.4}$$

Using the representation (see the Appendix)

$$S(t'',t_1'')=F(t'')+F(t_1'')-F(t''-t_1''), \tag{2.5}$$

where

$$F(z)=\int_0^{|z|} du\, r(u)[|z|-u], \tag{2.6}$$

we can rewrite Eq. (2.3) in the form

$$\sigma_x^2(t)=2\frac{e^{-\gamma t}-e^{-2\gamma t}}{m\lambda}\int_0^t dt'' F(t'') e^{\gamma t''}-\frac{e^{-2\gamma t}}{m^2}G(t). \tag{2.7}$$

The function $G(t)$ is defined as

$$G(t)=\int_0^t dt'' \int_0^t dt_1'' F(t''-t_1'') e^{\gamma(t''+t_1'')}. \tag{2.8}$$

With the transformation of variables $u=t'-t_1'$ and $v=t'+t_1'$, we can write it as follows:

$$G(t)=\frac{1}{2}\int_{-t}^t du F(u)\int_{|u|}^{2t-|u|} dv\, e^{\gamma v} = \frac{1}{\gamma}\int_0^t du F(u)[e^{\gamma(2t-u)}-e^{\gamma u}]. \tag{2.9}$$

Substituting Eq. (2.9) into Eq. (2.7), we obtain the desired expression

$$\sigma_x^2(t)=\frac{1}{m\lambda}\int_0^t du F(u)[2e^{-\gamma(t-u)}-e^{-\gamma(2t-u)}-e^{-\gamma u}], \tag{2.10}$$

which is much simpler than Eq. (2.3).

According to Eq. (2.2), the particle velocity is given by

$$v(t)=\dot{x}(t)=-\gamma x(t)+\frac{1}{m}\int_0^t dt' f(t'). \tag{2.11}$$

Its dispersion $\sigma_v^2(t)=\langle \dot{x}^2(t)\rangle$ can then be written in the form

$$\sigma_v^2(t)=\gamma^2\sigma_x^2(t)-\frac{2\gamma}{m}\int_0^t dt' \langle f(t')x(t)\rangle + \frac{1}{m^2}\left\langle\left(\int_0^t dt' f(t')\right)^2\right\rangle. \tag{2.12}$$

Using Eq. (2.10) and the relations

$$\left\langle\left(\int_0^t dt' f(t')\right)^2\right\rangle=2F(t), \tag{2.13}$$

$$\int_0^t dt' \langle f(t')x(t)\rangle=\frac{1}{m}\int_0^t du F(u)[e^{-\gamma(t-u)}-e^{-\gamma u}] + \frac{1-e^{-\gamma t}}{\lambda}F(t), \tag{2.14}$$

which follow from Eqs. (2.2) and (2.5), we obtain

$$\sigma_v^2(t)=\frac{\gamma}{m^2}\int_0^t du F(u)[e^{-\gamma u}-e^{-\gamma(2t-u)}]+\frac{2}{m^2}F(t)e^{-\gamma t}. \tag{2.15}$$

Finally, for the cross dispersion between the particle position and the particle velocity, $\sigma_{xv}(t)=\langle x(t)\dot{x}(t)\rangle =1/2d\sigma_x^2(t)/dt$, we find

$$\sigma_{xv}(t)=-\frac{1}{m^2}\int_0^t du F(u)[e^{-\gamma(t-u)}-e^{-\gamma(2t-u)}] + \frac{1-e^{-\gamma t}}{m\lambda}F(t). \tag{2.16}$$

Note that for a Gaussian driving force $f(t)$, the dispersions $\sigma_x^2(t)$, $\sigma_v^2(t)$, and $\sigma_{xv}(t)$ fully determine the probability density of the particle position $p(x,t)$, of the particle velocity $p(v,t)$, and the joint probability density of x and v, $p(x,v,t)$. In this case, $x(t)$ and $\dot{x}(t)$ are Gaussian processes with zero mean, and

$$p(x,t)=\frac{1}{\sqrt{2\pi}\sigma_x(t)}\exp\left(-\frac{x^2}{2\sigma_x^2(t)}\right), \tag{2.17}$$

$$p(v,t)=\frac{1}{\sqrt{2\pi}\sigma_v(t)}\exp\left(-\frac{v^2}{2\sigma_v^2(t)}\right), \tag{2.18}$$

$$p(x,v,t)=\exp\left(-\frac{\sigma_v^2(t)x^2-2\sigma_{xv}(t)xv+\sigma_x^2(t)v^2}{2[\sigma_x^2(t)\sigma_v^2(t)-\sigma_{xv}^2(t)]}\right) \times\frac{1}{2\pi\sqrt{\sigma_x^2(t)\sigma_v^2(t)-\sigma_{xv}^2(t)}}. \tag{2.19}$$

These probability densities obey the initial conditions $p(x,0)=\delta(x)$, $p(v,0)=\delta(v)$, and $p(x,v,0)=\delta(x)\delta(v)$, which are equivalent to Eq. (2.1).

III. ASYMPTOTIC BEHAVIOR

A. Damped particles $(0<\gamma<\infty)$

The long-time behavior of $\sigma_x^2(t)$, $\sigma_v^2(t)$, and $\sigma_{xv}(t)$ depends on certain properties of the noise correlation function $r(u)$. Equations (1.2), (2.3), and (2.13) imply that $r(0)\geqslant 0$, $\sigma_x^2(t)\geqslant 0$, and $F(u)\geqslant 0$. The condition $\sigma_x^2(t)\geqslant 0$ requires that $S(t'',t_1'')\geqslant 0$, which is the case if and only if $F(u)$ is a nondecreasing function of $u(\geqslant 0)$, i.e., $dF(u)/du\geqslant 0$ or $\int_0^u ds\, r(s)\geqslant 0$, as can be seen from Eq. (2.5). Since $F(0)=0$, the condition $F(u)\geqslant 0$ always holds for any nondecreasing function $F(u)$. We assume that the correlations between $f(t)$ and $f(t')$ decrease as $|t-t'|$ increases. Thus $r(\infty)=0$, and the function $F(u)$ increases slower than u^2, i.e., $\lim_{u\to\infty}F(u)/u^2=0$. These properties imply that for damped particles the asymptotic behavior of the velocity dispersion (2.15), at $t=\infty$ is given by

$$\sigma_v^2(\infty)=\frac{\gamma}{m^2}\int_0^\infty du F(u)e^{-\gamma u}. \tag{3.1}$$

The dispersion has a finite value that is proportional to the Laplace transform of $F(u)$. If the random driving force corresponds to internal equilibrium fluctuations, i.e., $r(u)=2\lambda k_B T\delta(u)$ where T is the absolute temperature and k_B is the Boltzmann constant, then Eq. (3.1) leads, as expected, to the law of equipartition of energy, i.e., $m\sigma_v^2(\infty)/2=k_BT/2$.

The asymptotic behavior of the position dispersion $\sigma_x^2(t)$ as $t\to\infty$, is governed by the asymptotic behavior of $F(u)$ as $u\to\infty$. There are two qualitatively different cases, namely, $0<F(\infty)<\infty$ and $F(\infty)=\infty$. [The case $F(\infty)=0$ is not realized.] The first case requires that

$$\int_0^u ds\, r(s)=o(1/u) \quad (u\to\infty), \tag{3.2}$$

and Eq. (2.10) then implies that $\sigma_x^2(t)$ tends to a finite limit as $t\to\infty$:

$$\sigma_x^2(\infty)=\frac{2}{\lambda^2}F(\infty)-\frac{1}{m\lambda}\int_0^\infty du F(u)e^{-\gamma u}. \tag{3.3}$$

We call this phenomenon stochastic localization, i.e., the mean-square displacement for free particles driven by correlated noise approaches a finite value, $\sigma_x^2(\infty)<\infty$, in the long-time limit. As expected on physical grounds, Eq. (3.2) shows that stochastic localization of particles is caused by negative correlations in the random force $f(t)$. Combining Eqs. (3.1) and (3.3), we obtain the relation

$$\frac{1}{2}m\sigma_v^2(\infty)+\frac{1}{2}\frac{\lambda^2}{m}\sigma_x^2(\infty)=\frac{1}{m}F(\infty). \tag{3.4}$$

The range of the particle localization, $\sigma_x(\infty)$, decreases as the damping coefficient λ increases. If $0<F(\infty)<\infty$, Eq. (3.4) reveals that the long-time behavior of a free particle is similar to the behavior of a harmonic oscillator having a stiffness coefficient λ^2/m and interacting with a heat bath having a temperature $T=F(\infty)/mk_B$. Equation (2.16) implies that the values $x(\infty)$ and $\dot{x}(\infty)$ are not correlated, i.e., $\sigma_{xv}(\infty)=0$.

In the second case, $F(\infty)=\infty$, Eq. (2.10) leads to the following asymptotic formula:

$$\sigma_x^2(t)\sim\frac{2}{\lambda^2}F(t) \quad \text{as} \quad t\to\infty. \tag{3.5}$$

The noise intensity R, defined as [36]

$$R=\int_0^\infty du\, r(u), \tag{3.6}$$

satisfies the condition $R\geqslant 0$, which follows from the condition $F(u)\geqslant 0$. This parameter characterizes both the asymptotic behavior of $r(u)$ as $u\to\infty$ and the influence of regions of negative correlations for which $r(u)<0$. Specifically, the value $R=\infty$ corresponds to the case of slowly decreasing positive correlations, i.e., $ur(u)\to\infty$ as $u\to\infty$. The value $R=0$ corresponds to the case where contributions from regions of positive and negative correlations balance. From Eqs. (3.5) and (2.6) we obtain that (i) $\sigma_x^2(t)$ diverges faster than t but slower than t^2 if $R=\infty$, that (ii) $\sigma_x^2(t)\sim 2Rt/\lambda^2$ if $0<R<\infty$, and that (iii) $\sigma_x^2(t)$ diverges slower than t if $R=0$. If $F(t)$ obeys a power law, $F(t)\sim t^\nu$, superdiffusion takes place for $R=\infty$, normal diffusion for $0<R<\infty$, and subdiffusion for $R=0$. The divergence of $ur(u)$ as $u\to\infty$ is a necessary, but not sufficient, condition for superdiffusion, and a sign change of $r(u)$ is a necessary, but not sufficient, condition for subdiffusion.

B. Undamped particles $(\gamma=0)$

The position, velocity, and cross dispersion for undamped particles are obtained from the general expressions in Sec. II by taking the limit $\gamma\to 0$. For the dispersion of the particle position, Eq. (2.10) yields

$$\sigma_x^2(t)=\frac{2}{m^2}\int_0^t du F(u)u. \tag{3.7}$$

Since $F(\infty)\neq 0$, $\sigma_x^2(t)$ always diverges as $t\to\infty$ for undamped particles. If $F(\infty)<\infty$, which implies that $R=0$, then $\sigma_x^2(t)\sim F(\infty)t^2/m^2$, and if $F(\infty)=\infty$, then $\sigma_x^2(t)$ diverges faster than t^2 but slower than t^4. In the latter case, Eq. (3.7) implies that (i) $\sigma_x^2(t)$ diverges faster than t^3 but slower than t^4 for $R=\infty$, that (ii) $\sigma_x^2(t)\sim 2Rt^3/3m^2$ for $0<R<\infty$, and that (iii) $\sigma_x^2(t)$ diverges faster than t^2 but slower than t^3 for $R=0$. In particular, if $f(t)$ is a white noise, i.e, $r(u)=2\Delta\delta(u)$, where Δ is the white noise intensity, then $F(u)=\Delta u$, and Eq. (3.7) yields $\sigma_x^2(t)=2\Delta t^3/3m^2$, which is valid for all times. If $F(u)$ obeys a power law, $F(u)\sim u^\nu$ as $u\to\infty$, then for undamped particles we have $\sigma_x^2(t)\sim t^{2+\nu}$, whereas for damped particles $\sigma_x^2(t)\sim t^\nu$. We suggest therefore that for undamped particles with $\sigma_x^2(t)\sim t^\eta$ as $t\to\infty$, the case $\eta=3$ should be called normal diffusion; the case $3<\eta<4$, superdiffusion; and the case $2<\eta<3$, subdiffusion.

Since stochastic localization is the limiting case of subdiffusion, the case $\eta=2$ should also be called subdiffusion.

As $\lambda\to 0$, Eq. (2.15) is reduced to

$$\sigma_v^2(t)=\frac{2}{m^2}F(t). \tag{3.8}$$

Therefore if $F(\infty)<\infty$, the dispersion of the particle velocity tends to a finite value, $\sigma_v^2(\infty)=2F(\infty)/m^2$, as $t\to\infty$. If $F(\infty)=\infty$, then $\sigma_v^2(t)$ diverges as $t\to\infty$. This means that the average kinetic energy of the particle, $E(t)=m\sigma_v^2(t)/2$, increases. The random driving force $f(t)$ leads to stochastic acceleration of the particle. In particular, for white noise, Eq. (3.8) yields $E(t)=\Delta t/m$, which is known as Fermi acceleration [37]. If $F(t)$, and consequently $E(t)$, obeys a power law, $E(t)\sim t^{\nu}$ as $t\to\infty$, then $1<\nu<2$ for $R=\infty$, $\nu=1$ for $0<R<\infty$, and $0<\nu<1$ for $R=0$. The slower the noise correlation function $r(u)$ decreases with u, the more effective the stochastic acceleration.

Finally, for undamped particles, Eq. (2.16) yields $\sigma_{xv}(t)=F(t)t/m^2$, and this quantity always diverges as $t\to\infty$.

C. Overdamped particles $(\gamma=\infty)$

For modeling purposes, it is often assumed that the damping coefficient λ is very large, so that the acceleration $\ddot{x}(t)$ becomes negligible. This approximation of overdamped particles corresponds to the limit $\gamma\to\infty$. In this limit we can simplify the integral in Eq. (2.10) using Laplace's method [38]. Taking into account that $F(0)=0$ and $F(\infty)\neq 0$, we obtain

$$\sigma_x^2(t)\sim\frac{2}{\lambda^2}F(t) \quad \text{as} \quad \gamma\to\infty. \tag{3.9}$$

Note that this formula is valid for all times, and it can be obtained directly from the equation of motion for overdamped particles, $\lambda\dot{x}(t)=f(t)$. Comparing Eqs. (3.9) and (3.5), we find that the long-time behavior of the dispersion of the particle position is the same for damped and overdamped particles, if and only if $F(\infty)=\infty$.

IV. TWO EXAMPLES OF RANDOM DRIVING FORCES

To illustrate the results obtained in the preceding section, we consider first a class of random forces $f(t)$ characterized by the correlation function

$$r(u)=r(0)\left(1+\frac{u}{\tau_0}\right)^{-\alpha} \quad \text{with} \quad \alpha>0. \tag{4.1}$$

For random forces in this class, we find that $R\neq 0$. Specifically, $R=\infty$ if $0<\alpha\leqslant 1$ and $0<R<\infty$ if $\alpha>1$. Straightforward calculations yield

$$F(u)=r(0)\tau_0^2\times\begin{cases}\dfrac{1}{(1-\alpha)(2-\alpha)}\left(1+\dfrac{u}{\tau_0}\right)^{2-\alpha}-\dfrac{1}{1-\alpha}\left(1+\dfrac{u}{\tau_0}\right)+\dfrac{1}{2-\alpha}, & \alpha\neq 1,2\\ \left(1+\dfrac{u}{\tau_0}\right)\ln\left(1+\dfrac{u}{\tau_0}\right)-\dfrac{u}{\tau_0}, & \alpha=1\\ \dfrac{u}{\tau_0}-\ln\left(1+\dfrac{u}{\tau_0}\right), & \alpha=2.\end{cases} \tag{4.2}$$

Since $F(\infty)=\infty$ for all α, the long-time behavior of $\sigma_x^2(t)$, both for damped and overdamped particles, is given by Eq. (3.5). Using Eq. (4.2), we find

$$\sigma_x^2(t)\sim\frac{2r(0)\tau_0^2}{\lambda^2}\times\begin{cases}\dfrac{1}{(1-\alpha)(2-\alpha)}\left(\dfrac{t}{\tau_0}\right)^{2-\alpha}, & 0<\alpha<1\\ \dfrac{t}{\tau_0}\ln\dfrac{t}{\tau_0}, & \alpha=1\\ \dfrac{1}{\alpha-1}\dfrac{t}{\tau_0}, & \alpha>1.\end{cases} \tag{4.3}$$

For the class of random forces with correlation function Eq. (4.1), there are three different regimes: (i) superdiffusion, if $0<\alpha<1$, (ii) power-logarithmic diffusion [$\sigma_x^2(t)\sim t^{\kappa}\ln t$], if $\alpha=1$, and (iii) normal diffusion, if $\alpha>1$. It is remarkable that for $\alpha=1$ the mean-square displacement grows as $t\ln t$, i.e., faster than normal diffusion but slower than superdiffusion. The same behavior was found in [33] for free particles with nonlocal dissipation. For undamped particles, Eqs. (3.7) and (4.2) also lead to three regimes as $t\to\infty$

$$\sigma_x^2(t) \sim \frac{2r(0)\tau_0^4}{m^2} \times \begin{cases} \frac{1}{(1-\alpha)(2-\alpha)(4-\alpha)}\left(\frac{t}{\tau_0}\right)^{4-\alpha}, & 0<\alpha<1 \\ \frac{1}{3}\left(\frac{t}{\tau_0}\right)^3 \ln\frac{t}{\tau_0}, & \alpha=1 \\ \frac{1}{3(\alpha-1)}\left(\frac{t}{\tau_0}\right)^3, & \alpha>1. \end{cases} \tag{4.4}$$

As mentioned above, the mean-square displacement grows t^2 times faster than for damped particles. According to the terminology suggested in Sec. III, the case $0<\alpha<1$ corresponds to superdiffusion, the case $\alpha=1$ to power-logarithmic diffusion, and the case $\alpha>1$ to normal diffusion. Note that the average kinetic energy of undamped particles grows as $E(t) \sim (\lambda^2/2m)\sigma_x^2(t)$ as $t\to\infty$ where $\sigma_x^2(t)$ is given by Eq. (4.3).

Next we consider a class of random forces $f(t)$ characterized by the correlation function

$$r(u) = r(0)\left(1+\frac{u}{\tau_0}\right)^{-\beta}\left(1-(\beta-2)\frac{u}{\tau_0}\right) \text{ with } \beta>2, \tag{4.5}$$

for which the condition $R=0$ holds. In this case,

$$F(u) = r(0)\tau_0^2 \times \begin{cases} \frac{1}{3-\beta}\left[\left(1+\frac{u}{\tau_0}\right)^{3-\beta}-1\right]-\frac{1}{2-\beta}\left[\left(1+\frac{u}{\tau_0}\right)^{2-\beta}-1\right], & \beta\neq 3 \\ \ln\left(1+\frac{u}{\tau_0}\right)+\left(1+\frac{u}{\tau_0}\right)^{-1}-1, & \beta=3, \end{cases} \tag{4.6}$$

and Eqs. (3.3) and (3.5) yield

$$\sigma_x^2(t) \sim \frac{2r(0)\tau_0^2}{\lambda^2} \times \begin{cases} \frac{1}{3-\beta}\left(\frac{t}{\tau_0}\right)^{3-\beta}, & 2<\beta<3 \\ \ln\frac{t}{\tau_0}, & \beta=3 \\ \frac{G_{\beta-3}(\gamma\tau_0)}{2(\beta-3)}-\frac{G_{\beta-2}(\gamma\tau_0)}{2(\beta-2)}, & \beta>3, \end{cases} \tag{4.7}$$

where $G_\sigma(y)=1+e^y y^\sigma \Gamma(1-\sigma,y)$ and $\Gamma(1-\sigma,y)=\int_y^\infty dz e^{-z} z^{-\sigma}$ is the incomplete gamma function. This class of random forces also gives rise to three different regimes for the long-time behavior of the mean-square displacement of damped particles: (i) subdiffusion, if $2<\beta<3$, (ii) logarithmic diffusion, if $\beta=3$, and (iii) stochastic localization, if $\beta>3$. For overdamped particles in the regime of stochastic localization we have $\sigma_x^2(\infty)=2r(0)\tau_0^2/\lambda^2(\beta-2)(\beta-3)$, and for undamped particles we obtain

$$\sigma_x^2(t) \sim \frac{2r(0)\tau_0^4}{m^2} \times \begin{cases} \frac{1}{(3-\beta)(5-\beta)}\left(\frac{t}{\tau_0}\right)^{5-\beta}, & 2<\beta<3 \\ \frac{1}{2}\left(\frac{t}{\tau_0}\right)^2 \ln\frac{t}{\tau_0}, & \beta=3 \\ \frac{1}{2(\beta-2)(\beta-3)}\left(\frac{t}{\tau_0}\right)^2, & \beta>3. \end{cases} \tag{4.8}$$

The cases $2<\beta<3$ and $\beta>3$ correspond to subdiffusion, and the case $\beta=3$, to power-logarithmic diffusion.

V. CONCLUSIONS

We have used the Langevin approach to describe free damped particles driven by an arbitrary stationary noise and to obtain exact analytical results for the dispersion of the particle position, of the particle velocity, and their cross dispersion. On the basis of those results, we have studied in detail the influence of noise correlations on the character of particle diffusion. The relevant parameter to characterize the influence of the noise correlations is the noise intensity R, i.e., the integral of the correlation function $r(t)$.

We have shown that the condition $0<R<\infty$ corresponds to normal diffusion, and the conditions $R=\infty$ and $R=0$ correspond to anomalous diffusion. If $R=\infty$, diffusion is faster than normal diffusion, and if $R=0$ it is slower. The spectral density $S(\omega)$ of the (wide-sense) stationary random force $f(t)$ is given by

$$S(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}r(t)dt=2\int_{0}^{\infty}r(t)\cos(\omega t)dt. \quad (5.1)$$

Therefore, $S(0)=2R$, and external noise with very strong coherence, $S(\omega)\to\infty$ as $\omega\to 0$, gives rise to diffusion faster than normal, whereas external noise with very weak coherence, $S(\omega)\to 0$ as $\omega\to 0$, leads to diffusion slower than normal. In the case of fast diffusion, as $t\to\infty$, the mean-square displacement $\sigma_x^2(t)$ grows faster than t but slower than t^2, and in the case of slow diffusion slower than t. The conditions $R=\infty$ and $R=0$ are necessary but not sufficient for the existence of superdiffusion and subdiffusion, respectively. Specifically, for $R=\infty$ power-logarithmic diffusion can occur and for $R=0$ logarithmic diffusion. Our results show that a long time-tail decay of positive correlations of the noise gives rise to fast diffusion, and a balance of contributions from regions of positive and negative correlations to slow diffusion. Also, if $R=0$, stochastic localization of particles can occur, where $\sigma_x^2(t)$ is finite for all times.

For undamped particles, diffusion is t^2 times faster than for damped particles. The condition $\sigma_x^2(t)\sim t^3$ as $t\to\infty$ corresponds to normal diffusion of undamped particles. Fast diffusion occurs if $R=\infty$, and slow diffusion if $R=0$, as in the case of damped particles. The dispersion $\sigma_x^2(t)$, as $t\to\infty$, grows faster than t^3 but slower than t^4 for fast diffusion, and faster than t^2 but slower than t^3 for slow diffusion.

APPENDIX: DERIVATION OF EQ. (2.5)

By introducing the variables $u=t''-t_1''$ and $v=t''+t_1''$, we can write Eq. (2.4) for $t''\geqslant t_1''$ as

$$\begin{aligned}S(t'',t_1'')=&\ \tfrac{1}{2}\int_{-t_1''}^{0}du\, r(u)\int_{-u}^{2t''+u}dv\\ &+\tfrac{1}{2}\int_{0}^{t''-t_1''}du\, r(u)\int_{u}^{2t_1''+u}dv\\ &+\tfrac{1}{2}\int_{t''-t_1''}^{t''}du\, r(u)\int_{u}^{2t''-u}dv \quad (A1)\end{aligned}$$

$$\begin{aligned}=&\int_0^{t_1''}du\, r(u)(t_1''-u)+\int_0^{t''-t_1''}du\, r(u)t_1''\\ &+\int_{t''-t_1''}^{t''}du\, r(u)(t''-u) \quad (A2)\end{aligned}$$

$$\begin{aligned}=&\int_0^{t_1''}du\, r(u)(t_1''-u)+\int_0^{t''}du\, r(u)(t''-u)\\ &-\int_0^{t''-t_1''}du\, r(u)(t''-t_1''-u). \quad (A3)\end{aligned}$$

Taking into account the definition (2.6), we obtain Eq. (2.5) from Eq. (A3). Equation (2.5) also holds for $t''<t_1''$, since we have $S(t'',t_1'')=S(t_1'',t'')$ according to Eq. (2.4).

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